

Preprint typeset in JHEP style - HYPER VERSION

AEI-2006-071

HU-EP-06/31

ITP-UU-06-39

SPIN-06-33

TCDMATH 06-13

The Off-shell Symmetry Algebra of the Light-cone $\text{AdS}_5 \times \text{S}^5$ Superstring

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ABSTRACT: We analyze the $\mathfrak{psu}(2, 2|4)$ supersymmetry algebra of a superstring propagating in the $\text{AdS}_5 \times \text{S}^5$ background in the uniform light-cone gauge. We consider the off-shell theory by relaxing the level-matching condition and take the limit of infinite light-cone momentum, which decompactifies the string world-sheet. We focus on the $\mathfrak{psu}(2|2) \oplus \mathfrak{psu}(2|2)$ subalgebra which leaves the light-cone Hamiltonian invariant and show that it undergoes extension by a central element which is expressed in terms of the level-matching operator. This result is in agreement with the conjectured symmetry algebra of the dynamic S-matrix in the dual $\mathcal{N} = 4$ gauge theory.

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Contents

1. Introduction	1
2. Gauged-fixed string sigma-model	4
3. Symmetry generators in the light-cone gauge	7
3.1 General structure of the symmetry generators	7
3.2 Moment map and the Poisson brackets	9
3.3 Explicit basis	12
4. Deriving the central charges	13
5. Concluding remarks	17
6. Appendix	18
6.1 The gauge-fixed Hamiltonian	18
6.2 Symmetry generators	20
6.3 Covariant notation	21
6.4 The explicit form of the $\mathfrak{su}(2 2)_R$ generators	23
6.5 The $\mathfrak{su}(2 2)_R$ supercharge at quartic field order	24
6.6 The explicit form of the $\mathfrak{su}(2 2)_L$ generators	24
6.7 The centrality of the level-matching and Hamiltonian	26

1. Introduction

Understanding the symmetries of physical systems usually leads to the most elegant way of solving them. The Green-Schwarz string theory on the $\text{AdS}_5 \times S^5$ background [1] presents a prime example of a system with a very large number of symmetries. The manifest global symmetry of the string sigma-model is given by the $\text{PSU}(2, 2|4)$ supergroup, an isometry group of the target coset space. Moreover the world-sheet theory is a classically integrable model [2], possessing thus an infinite number of (non)local integrals of motion. Finally, being a superstring theory, it possesses the local gauge symmetries of world-sheet diffeomorphisms and the fermionic κ -symmetry.

Nevertheless, the quantisation of the superstring on $\text{AdS}_5 \times S^5$ in a covariant fashion is presently out of reach. However, fixing the world-sheet gauge symmetry and reducing the theory to only physical degrees of freedom simplifies the system

and allows one to invoke the notion of asymptotic states and the machinery of the string S-matrix [3, 4, 5]. Based on extensive work [6]–[11] it is becoming more and more apparent that a particularly convenient world-sheet gauge choice is the so-called *generalised uniform light-cone gauge* [8, 10]. World-sheet excitations in this gauge are closely related to the spin chain excitation picture appearing on the gauge theory side, and thus make the link between the two more transparent. Also, similarly as in flat space, this is the only gauge which makes the Green-Schwarz fermions tractable.

The generalised uniform light-cone gauge, however, has the unpleasant feature that the gauge-fixed action does not possess world-sheet Lorentz invariance. The fact that this symmetry is absent makes the construction of the world-sheet S-matrix particularly involved: standard constraints on the form of S-matrix arising from the requirement of Lorentz invariance cannot be directly implemented, but require subtle constructions [12, 13]. It is thus extremely important to understand the symmetries of the world-sheet theory in this gauge, how they constrain the form of the S-matrix, and how they are connected with the symmetries present in the gauge theory.

The generators of the supersymmetry algebra $\mathfrak{psu}(2, 2|4)$ of the string sigma-model can be split into two groups: those which (Poisson) commute with the Hamiltonian and those which do not. The former comprise the $\mathfrak{su}(2|2) \oplus \mathfrak{su}(2|2)$ subalgebra of the full $\mathfrak{psu}(2, 2|4)$ algebra, sharing the same central element which corresponds to the Hamiltonian. Another separation of the $\mathfrak{psu}(2, 2|4)$ generators is into dynamical and kinematical generators, depending on whether they do or do not depend on the unphysical field x_- .

There are three important facts related to the presence of the unphysical field x_- in the light-cone world-sheet theory. One is that when solved in terms of physical fields, x_- is a *non-local expression*. Second is that the zero mode of the field x_- is a priori non-zero and has a non-trivial Poisson bracket with the total light-cone momentum P_+ . This implies that dynamical charges change the light-cone momentum P_+ . It also follows, that the zero mode part of the operator $e^{i\alpha x_-}$ plays precisely the role of a “length changing” operator, given that in the uniform light cone gauge, the total light cone momentum P_+ is naturally identified with the circumference of the world-sheet cylinder. However, and this shall be exploited extensively in the present paper, in the limit of infinite light-cone momentum P_+ , the fluctuations of the P_+ are irrelevant, and the zero mode of x_- can be thus consistently ignored. Thirdly, the differentiated field x'_- is a density of the world-sheet momentum related to the presence of rigid symmetries in the space-like σ -direction of the light-cone gauge fixed string action. In the case of closed strings, the periodicity of fields implies that the total world-sheet momentum p_{ws} has to vanish. This constraint is called the level-matching condition.

In order to introduce the concept of the world-sheet excitations as well as the

world-sheet S-matrix, one needs to: (a) relax the level-matching condition and (b) consider the limit $P_+ \rightarrow \infty$. If level-matching is not satisfied, we will refer to the theory as “off-shell”. The limit (b) is necessary in order to define asymptotic states and it basically defines the gauge fixed world-sheet theory on the plane, rather than on a cylinder of circumference P_+ [14]-[20]. In this paper, we will restrict the consideration to this limit, only briefly commenting on the finite P_+ configurations in the discussion.

If the level-matching condition holds, the $\mathfrak{su}(2|2) \oplus \mathfrak{su}(2|2)$ subalgebra of the $\mathfrak{psu}(2, 2|4)$ algebra is spanned by those generators which commute with the world-sheet Hamiltonian. Giving up the level-matching condition in string theory in principle could spoil the on-shell $\mathfrak{psu}(2, 2|4)$ symmetry algebra of the world-sheet theory and in particular the centrality of the Hamiltonian with respect to the $\mathfrak{su}(2|2) \oplus \mathfrak{su}(2|2)$ subalgebra. The main goal of this paper is the derivation of the string world-sheet symmetry algebra in the case when the level-matching is relaxed.

In $\mathcal{N} = 4$ gauge theory the analogue of the level-matching condition is implemented by considering gauge-invariant operators, i.e. traces of products of fields. Hence relaxing the level-matching in string theory corresponds to “opening” the trace of fields in gauge theory, i.e. considering open rather than closed spin chains. Beisert has argued in [5] that opening the traces in gauge theory (and considering infinitely long operators) leads to a modification of the $\mathfrak{su}(2|2) \oplus \mathfrak{su}(2|2)$ algebra: the algebra receives two central charges in addition to the Hamiltonian, which are functions of the momentum carried by the one-particle excitations. The Hamiltonian remains a central element. This centrally extended symmetry algebra beautifully allowed for the derivation of the dispersion relation of the elementary “magnon” excitations as well as for the restriction of the form of S-matrix down to one unknown function.

The main result of this paper is the *derivation* of the centrally extended $\mathfrak{su}(2|2) \oplus \mathfrak{su}(2|2)$ algebra in *string theory* both at the classical and quantum level. By explicit computations, we show that relaxing the level-matching condition in string theory in the limit of infinite light-cone momentum necessarily leads to an enlargement of the $\mathfrak{su}(2|2) \oplus \mathfrak{su}(2|2)$ algebra by a common central element which is proportional to the level-matching condition. In addition, the Hamiltonian remains central, as it was the case for the on-shell algebra.

The direct evaluation of the classical (and quantum) $\mathfrak{su}(2|2) \oplus \mathfrak{su}(2|2)$ algebra is technically very difficult due to the complexity of the supersymmetry generators and the non-canonical Poisson structure of the theory. To derive the central charges, we were thus forced to work in an approximation scheme which we named the “hybrid” expansion scheme. Namely, in this approximation we expand all supersymmetry generators in powers of fields (equivalently in the inverse string tension $2\pi/\sqrt{\lambda}$), keeping however all dependence on the x_- field *intact* and rigid. More precisely,

the dynamical charges depend on the x_- field via $e^{i\alpha x_-}$ in a multiplicative fashion. Although, when expressed in terms of the physical fields this term is highly non-linear, in the “hybrid” expansion scheme we treat this quantity as a single object. This allows us to determine *the full, non-linear, bosonic part* of the central charges. The fermionic part is then uniquely fixed from the requirement that the central charges vanish on the level-matching constraint surface. Justification for the “hybrid” expansion scheme is demonstrated in section 4.

2. Gauged-fixed string sigma-model

In this section we collect the necessary background material concerning the superstring theory on $\text{AdS}_5 \times \text{S}^5$. The central object on which the construction of the string action is based on is the well-known supersymmetry group $\text{PSU}(2, 2|4)$. We recall [1, 21, 22] that the superstring action S is a sum of two terms: the (world-sheet metric-dependent) kinetic term and the topological Wess-Zumino term:

$$S = -\frac{\sqrt{\lambda}}{4\pi} \int_{-r}^r d\sigma d\tau \left(\gamma^{\alpha\beta} \text{str}(\mathbf{A}_\alpha^{(2)} \mathbf{A}_\beta^{(2)}) + \kappa \epsilon^{\alpha\beta} \text{str}(\mathbf{A}_\alpha^{(1)} \mathbf{A}_\beta^{(3)}) \right), \quad (2.1)$$

Here $\frac{\sqrt{\lambda}}{2\pi}$ is the effective string tension, coordinates σ and τ parametrize the string world-sheet, and for later convenience we choose the range of σ to be $-r \leq \sigma \leq r$, where r is an arbitrary constant. The standard choice for a closed string is $r = \pi$. Then, $\gamma^{\alpha\beta} = \sqrt{-h} h^{\alpha\beta}$ where $h^{\alpha\beta}$ is the world-sheet metric, and $\kappa = \pm 1$ to guarantee the invariance of the action w.r.t. to the local κ -symmetry transformations. For the sake of clarity we choose in the rest of the paper $\kappa = +1$. Finally, $\mathbf{A}^{(i)}$ with $i = 0, \dots, 3$ denote the corresponding \mathbb{Z}_4 -projections of the flat current $\mathbf{A} = -g^{-1}dg$, where g is a representative of the coset space

$$\frac{\text{PSU}(2, 2|4)}{\text{SO}(4, 1) \times \text{SO}(5)}.$$

The above-described Lagrangian formulation does not seem to be a convenient starting point for studying many interesting properties of the theory, in particular, for analyzing its symmetry algebra and developing a quantization procedure, because it suffers from the presence of non-physical bosonic and fermionic degrees of freedom related to reparametrization and κ -symmetry transformations. A natural way to overcome this difficulty is to use the Hamiltonian formulation of the model. It is obtained by fixing the gauge symmetries and solving the Virasoro constraints, the latter arise as equations of motion for the world-sheet metric $h^{\alpha\beta}$. Concerning the quantization, it should be implemented in such a way that the global supersymmetry algebra, $\mathfrak{psu}(2, 2|4)$, which includes the Hamiltonian, being restricted to physical

states satisfying the level-matching condition would remain non-anomalous at the quantum level. Hopefully, the quantum Hamiltonian could be uniquely determined in this way and then the remaining problem would be to determine its spectrum.

A suitable gauge which leads to the removal of non-physical degrees of freedom has been introduced in [20], following earlier work in [23, 24, 7, 8, 10]. We refer to it as the generalized uniform light-cone gauge. To impose the generalized uniform light-cone gauge we make use of the global AdS time, t , and an angle ϕ which parametrizes one of the big circles of S^5 . They parametrize two $U(1)$ isometry directions of $AdS_5 \times S^5$, and the corresponding conserved charges, the space-time energy E and the angular momentum J , are related to the momenta conjugated to t and ϕ as follows

$$E = -\frac{\sqrt{\lambda}}{2\pi} \int_{-r}^r d\sigma p_t, \quad J = \frac{\sqrt{\lambda}}{2\pi} \int_{-r}^r d\sigma p_\phi.$$

Then we introduce the light-cone variables

$$x_+ = (1-a)t + a\phi, \quad x_- = \phi - t$$

whose definition involves one additional gauge parameter a : $0 \leq a \leq 1$. The corresponding canonical momenta are

$$p_- = p_t + p_\phi, \quad p_+ = (1-a)p_\phi - ap_t.$$

The reparametrization invariance is then used to fix the generalized uniform light-cone gauge by requiring that¹

$$x_+ = \tau, \quad p_+ = 1. \quad (2.2)$$

The consistency of this gauge choice requires the constant r to be

$$r = \frac{\pi}{\sqrt{\lambda}} P_+ \equiv \frac{1}{2} \int_{-r}^r d\sigma p_+, \quad (2.3)$$

where P_+ is the total light-cone momentum.²

Solving the Virasoro constraints, one is then left with 8 transverse coordinates x_M and their conjugate momenta p_M .

The light-cone gauge should be supplemented by a suitable choice of a κ -symmetry gauge which removes 16 out of 32 fermions from the supergroup element g parametrizing the coset $\frac{PSU(2,2|4)}{SO(4,1) \times SO(5)}$. The remaining 16 fermions χ have a highly non-trivial

¹Strictly speaking, x_+ can be identified with the world-sheet time τ only for string configurations with vanishing winding number. For the general consideration, see [10, 20].

²In fact, P_+ has a conjugate variable $x_-^{(0)}$ that is the zero mode of the light-cone coordinate x_- . In any set of functions which do not contain x_- the variable P_+ plays the role of a central element and, therefore, can be fixed to be a constant.

Poisson structure which can be reduced to the canonical one by performing an appropriate non-linear field redefinition, see [11] for detail.

The gauged-fixed action that follows from (2.1) upon fixing (2.2) and the κ -symmetry was obtained in [11], c.f. also [20]. Schematically, it has the following structure

$$S = \frac{\sqrt{\lambda}}{2\pi} \int_{-r}^r d\sigma d\tau \left(p_M \dot{x}_M + \chi^\dagger \dot{\chi} - \mathcal{H} \right), \quad (2.4)$$

where \mathcal{H} is the Hamiltonian density which is independent of both λ and P_+ , and is equal to $-p_-$. Since $p_- = p_t + p_\phi$, the world-sheet Hamiltonian is given by the difference of the space-time energy E and the U(1) charge J :

$$H = -\frac{\sqrt{\lambda}}{2\pi} \int_{-r}^r d\sigma p_- = E - J. \quad (2.5)$$

Some comments are in order. As standard to the light-cone gauge choice, the light-cone charge P_+ plays effectively the role of the length of the string: in eq.(2.4) the dependence on P_+ occurs in the integration bounds only. Thus, the limit $P_+ \rightarrow \infty$ defines the theory on a plane and, by this reason, it can be called the “decompactifying limit” [14]-[20]. Obviously, for the theory on a plane one should also specify the boundary conditions for physical fields. As usual in soliton theory [25], several choices of the boundary conditions are possible: the rapidly decreasing case, the case of finite-density, etc. In what follows we will consider the case of fields rapidly decreasing at infinity and show that it is this case which leads to the realization of the centrally extended $\mathfrak{su}(2|2) \oplus \mathfrak{su}(2|2)$ symmetry algebra.

In the particular case $a = 0$ we deal with the temporal gauge $t = \tau$ analyzed in [8] which implies that the light-cone charge P_+ coincides with the angular momentum J . In the rest of the paper we will be mostly concentrated with the symmetric choice $a = \frac{1}{2}$ studied in [10, 11]. The reason is that as was shown in those papers in this gauge the Poisson structure of fermions simplifies drastically, and that makes it easier to compute the Poisson algebra of the global symmetry charges. All results we obtain, however, are also valid for the general a light-cone gauge.

In what follows we find it convenient to use the inverse string tension $\zeta = \frac{2\pi}{\sqrt{\lambda}}$, and to rescale bosons $(p_M, x_M) \rightarrow \sqrt{\zeta}(p_M, x_M)$ and fermions $\chi \rightarrow \sqrt{\zeta}\chi$ in order to ensure the canonical Poisson brackets for the physical fields. Upon these redefinitions the Hamiltonian (2.5) admits the following expansion in powers of ζ

$$H = \int_{-r}^r d\sigma \left(\mathcal{H}_2 + \zeta \mathcal{H}_4 + \dots \right). \quad (2.6)$$

Here the leading term \mathcal{H}_2 is quadratic in physical fields, and \mathcal{H}_4 is quartic, and so on. Thus, ζ^{n-1} will be multiplied by \mathcal{H}_{2n} containing the product of $2n$ fields, and the

expansion in ζ is an expansion in the number of fields. The explicit expressions for \mathcal{H}_2 and \mathcal{H}_4 were derived in [11] and we also present them in the Appendix to make the paper self-contained.

To conclude this section we remark that the light-cone gauge does not allow one to completely remove *all* unphysical degrees of freedom. There is a non-linear constraint, known as the level-matching condition, which is left unsolved. This constraint is just the statement that the total world-sheet momentum of the closed string vanishes, and it reads in our case as³

$$p_{\text{ws}} = -\zeta \int_{-r}^r d\sigma \left(p_M x'_M - \frac{i}{2} \text{str}(\Sigma_+ \chi \chi') \right) = 0. \quad (2.7)$$

The variable p_{ws} generates rigid shifts in σ and, therefore, in the limit $P_+ \rightarrow \infty$ becomes a momentum generator on the plane. We come to the discussion of the influence of the level-matching constraint on the supersymmetry algebra in the next section.

3. Symmetry generators in the light-cone gauge

In this section we study the general structure of the global symmetry generators in the light-cone gauge and identify the subalgebra of symmetries of the gauge-fixed Hamiltonian. We also reformulate a problem of computing the Poisson brackets of symmetry generators in terms of the standard notions of symplectic geometry.

3.1 General structure of the symmetry generators

The Lagrangian (2.1) is invariant w.r.t. the global action of the symmetry group $\text{PSU}(2, 2|4)$. The generators of the Lie superalgebra $\mathfrak{psu}(2, 2|4)$ are realized by the corresponding Noether charges which comprise⁴ an 8×8 supermatrix Q . As was shown in [11], in the light-cone gauge the matrix Q can be schematically written as follows

$$Q = \int_{-r}^r d\sigma \Lambda U \Lambda^{-1}. \quad (3.1)$$

An explicit form of the matrix U can be found in [11] and also in appendix 6.2, formula (6.10). It is important to note here that U depends on physical fields (x, p, χ) but not on x_{\pm} . The only dependence of Q on x_{\pm} occurs through the matrix Λ which is of the form

$$\Lambda = e^{\frac{i}{2}x_+\Sigma_+ + \frac{i}{4}x_-\Sigma_-}, \quad (3.2)$$

³See Appendix for the notations.

⁴As explained in [26], the part of Q which is proportional to the identity matrix is not a generator of $\mathfrak{psu}(2, 2|4)$ and, therefore, it should be factored out.

where Σ_{\pm} are the diagonal matrices of the form

$$\Sigma_{\pm} = \text{diag}\left(\pm 1, \pm 1, \mp 1, \mp 1; 1, 1, -1, -1\right). \quad (3.3)$$

We recall that the field x_- is unphysical and can be solved in terms of physical excitations through the equation

$$x'_- = -\zeta\left(p_M x'_M - \frac{i}{2}\text{str}(\Sigma_+ \chi \chi')\right). \quad (3.4)$$

Linear combinations of components of the matrix Q produce charges which generate rotations, dilatation, supersymmetry and so on. To single them out one should multiply Q by a corresponding 8×8 matrix \mathcal{M} , and take the supertrace

$$Q_{\mathcal{M}} = \text{str}(Q\mathcal{M}). \quad (3.5)$$

In particular, the diagonal and off-diagonal 4×4 blocks of \mathcal{M} single out bosonic and fermionic charges of $\mathfrak{psu}(2, 2|4)$, respectively.

Depending on the choice of \mathcal{M} the charges $Q_{\mathcal{M}} \equiv Q_{\mathcal{M}}(x_+, x_-)$ can be naturally classified according to their dependence on x_{\pm} . Firstly, with respect to x_- they are divided into *kinematical* (independent of x_-) and *dynamical* (dependent on x_-). Kinematical generators do not receive quantum corrections, while the dynamical generators do. Secondly, the charges, both kinematical and dynamical, may or may not depend on $x_+ = \tau$.

In the Hamiltonian setting the conservation laws have the following form

$$\frac{dQ_{\mathcal{M}}}{d\tau} = \frac{\partial Q_{\mathcal{M}}}{\partial \tau} + \{H, Q_{\mathcal{M}}\} = 0.$$

Therefore, the generators independent of $x_+ = \tau$ Poisson-commute with the Hamiltonian. As follows from the Jacobi identity, they must form an algebra which contains H as the central element.

Analyzing the structure of Q one can establish how a generic matrix \mathcal{M} is split into 2×2 blocks each of them giving rise either to kinematical or dynamical generators. This splitting of \mathcal{M} is shown in Figure 1, where the kinematical blocks are denoted by \mathbf{k} and the dynamical ones by \mathbf{d} respectively. Furthermore, one can see that the blocks which are colored in red and blue give rise to charges which are independent of $x_+ = \tau$; by this reason these charges commute with the Hamiltonian. Complementary, we note that the uncolored both kinematical (fermionic) and dynamical (bosonic) generators do depend on x_+ .

These conclusions about the structure of \mathcal{M} can be easily drawn by noting that Λ in eq.(3.2) is built out of two commuting matrices Σ_+ and Σ_- (see (3.3)). For instance, leaving in \mathcal{M} the kinematical blocks only, i.e. $\mathcal{M} \equiv \mathcal{M}_{\text{kin}}$, we find that

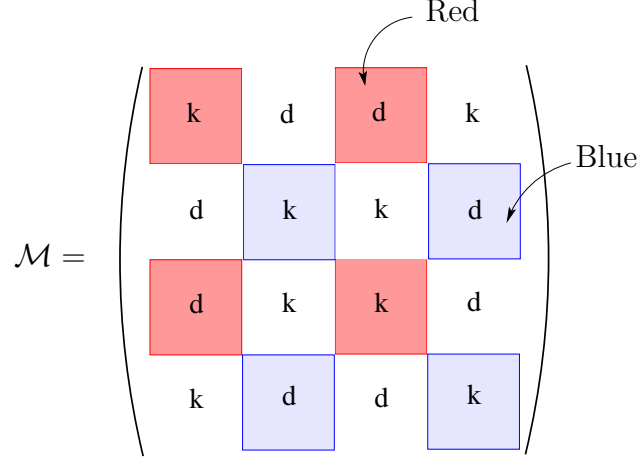


Figure 1: The distribution of the kinematical and dynamical charges in the \mathcal{M} supermatrix. The red (dark) and blue (light) blocks correspond to the subalgebra \mathcal{J} of $\mathfrak{psu}(2,2|4)$ which leaves the Hamiltonian invariant.

$[\Sigma_-, \mathcal{M}_{\text{kin}}] = 0$ and, therefore, due to the structure of $Q_{\mathcal{M}}$, see eq.(3.5), the variable x_- cancels out in $Q_{\mathcal{M}}$. Explicitly one finds the following conjugation expressions with Λ of (3.2)

$$\begin{aligned} \Lambda^{-1} \mathcal{M}_{\text{dyn}}^{\text{odd}} \Lambda &= e^{-\frac{i}{2}x_- \Sigma_-} \mathcal{M}_{\text{dyn}}^{\text{odd}}, & \Lambda^{-1} \mathcal{M}_{\text{dyn}}^{\text{even}} \Lambda &= \Lambda^2 \mathcal{M}_{\text{dyn}}^{\text{even}}, \\ \Lambda^{-1} \mathcal{M}_{\text{kin}}^{\text{odd}} \Lambda &= e^{i x_+ \Sigma_+} \mathcal{M}_{\text{kin}}^{\text{odd}}, & \Lambda^{-1} \mathcal{M}_{\text{kin}}^{\text{even}} \Lambda &= \mathcal{M}_{\text{kin}}^{\text{even}} \end{aligned}$$

showing that the $x_+ = \tau$ independent matrices are indeed given by $\mathcal{M}_{\text{dyn}}^{\text{odd}}$ and $\mathcal{M}_{\text{kin}}^{\text{even}}$, i.e. by the red and blue entries in Figure 1.

Finally, we note that the Hamiltonian itself can be obtained from Q as follows

$$H = -\frac{i}{2} \text{str}(Q \Sigma_+). \quad (3.6)$$

Another integral of motion, P_+ , is given by

$$P_+ = \frac{i}{4} \text{str}(Q \Sigma_-). \quad (3.7)$$

The structure of Q discussed above is found for finite r and it also remains valid in the limit $r \rightarrow \infty$.

3.2 Moment map and the Poisson brackets

The group $\text{PSU}(2,2|4)$ acts on the coset space $\frac{\text{PSU}(2,2|4)}{\text{SO}(4,1) \times \text{SO}(5)}$ by multiplications of a coset element by an element of the group from the left. When we fix the light-cone gauge and solve the Virasoro constraints we obtain the well-defined symplectic structure ω (the inverse of the Poisson bracket) for physical fields. Therefore, we

are now able to study the Poisson algebra of the Noether charges corresponding to the infinitesimal global symmetry transformations generated by the Lie algebra $\mathfrak{psu}(2, 2|4)$. Primarily we are interested in those charges which leave the gauge-fixed Hamiltonian and, as the consequence, the symplectic structure of the theory invariant; the corresponding subspace in $\mathfrak{psu}(2, 2|4)$ will be called \mathcal{J} .

Since the symplectic form ω remains invariant under the action of \mathcal{J} , to every element $\mathcal{M} \in \mathcal{J}$ one can associate a locally Hamiltonian phase flow $\xi_{\mathcal{M}}$ whose Hamiltonian function is nothing else but the Noether charge $Q_{\mathcal{M}}$:

$$\omega(\xi_{\mathcal{M}}, \dots) + dQ_{\mathcal{M}} = 0. \quad (3.8)$$

Identifying $\mathfrak{psu}(2, 2|4)$ with its dual space, $\mathfrak{psu}(2, 2|4)^*$, by using the supertrace operation, we can treat the matrix Q as the *moment map* [27] which maps the phase space (x, p, χ) into the dual space to the Lie algebra:

$$Q : (x, p, \chi) \rightarrow \mathfrak{psu}(2, 2|4)^*$$

and it allows one to associate to any element of $\mathfrak{psu}(2, 2|4)$ a function $Q_{\mathcal{M}}$ on the phase space. This linear mapping from the Lie algebra into the space of functions on the phase space is given by eq.(3.5). The function $Q_{\mathcal{M}}$ appears to be a Hamiltonian function, i.e. it obeys eq.(3.8), only if $\mathcal{M} \in \mathcal{J}$. Although the elements of $\mathfrak{psu}(2, 2|4)$ which do not belong to \mathcal{J} are the symmetries of the gauge-fixed action, they leave invariant neither the Hamiltonian nor the symplectic structure.

As is well known [28, 29], eq.(3.8) implies the following general formula for the Poisson bracket of the Noether charges $Q_{\mathcal{M}}$

$$\{Q_{\mathcal{M}_1}, Q_{\mathcal{M}_2}\} = (-1)^{\pi(\mathcal{M}_1)\pi(\mathcal{M}_2)} \text{str}(Q[\mathcal{M}_1, \mathcal{M}_2]) + C(\mathcal{M}_1, \mathcal{M}_2), \quad (3.9)$$

where $\mathcal{M}_{1,2} \in \mathcal{J}$. Here π is the parity of a supermatrix and $[\mathcal{M}_1, \mathcal{M}_2]$ is the graded commutator, i.e. it is the anti-commutator if both \mathcal{M}_1 and \mathcal{M}_2 are odd matrices, and the commutator if at least one of them is even. The first term in the r.h.s. of eq.(3.9) reflects the fact that the Poisson bracket of the Noether charges $Q_{\mathcal{M}_1}$ and $Q_{\mathcal{M}_2}$ gives a charge corresponding to the commutator $[\mathcal{M}_1, \mathcal{M}_2]$. The normalization prefactor $(-1)^{\pi(\mathcal{M}_1)\pi(\mathcal{M}_2)}$ is of no great importance and, as we will see later on, it is related to our specific choice of normalizing the even elements with respect to the odd ones inside the matrix Q . The quantity $C(\mathcal{M}_1, \mathcal{M}_2)$ in the r.h.s. of eq.(3.9) is the central extension, i.e. a bilinear graded skew-symmetric form on the Lie algebra \mathcal{J} which Poisson-commutes with all $Q_{\mathcal{M}}$, $\mathcal{M} \in \mathcal{J}$. The Jacobi identity for the bracket (3.9) implies that $C(\mathcal{M}_1, \mathcal{M}_2)$ is a two-dimensional cocycle of the Lie algebra \mathcal{J} . For simple Lie algebras such a cocycle necessarily vanishes, while for super Lie algebras it is generally not the case. Since we consider a finite-dimensional super Lie algebra the central extension vanishes if the element \mathcal{M} is bosonic: $C(\mathcal{M}, \dots) = 0$.

Some comments are necessary here. As we already mentioned in section 2, the usual feature of closed string theory considered in the light-cone gauge is the presence of the level-matching constraint $p_{\text{ws}} = 0$. This constraint arises from the requirement of the unphysical field x_- to be a periodic function of the world-sheet coordinate σ . The level-matching constraint cannot be solved in classical theory, rather it is required to vanish on physical states. Thus, before we turn our attention to the question of the physical spectrum we should treat p_{ws} as a non-trivial variable. We will refer to a theory with a non-vanishing generator p_{ws} as the *off-shell* theory. Since the Hamiltonian contains only physical fields it commutes with p_{ws} : $\{H, p_{\text{ws}}\} = 0$, i.e. the momentum p_{ws} is an integral of motion. The Poisson bracket (3.9) with the vanishing central term is valid on-shell and it is the off-shell theory where one could expect the appearance of a non-trivial central extension. Below we determine a general form of the central extension based on symmetry arguments only and in the next section by explicit evaluation of the Poisson brackets we justify the formula (3.9) and also find a concrete realization of $C(\mathcal{M}_1, \mathcal{M}_2)$.

Let us note that a formula as (3.9) makes it easy to reobtain our results on the structure of \mathcal{J} . Indeed, from eq.(3.9) we find that the invariance subalgebra $\mathcal{J} \subset \mathfrak{psu}(2, 2|4)$ of the Hamiltonian is determined by the condition

$$\{H, Q_{\mathcal{M}}\} = \text{str}(Q[\Sigma_+, \mathcal{M}]) = 0.$$

Thus, \mathcal{J} is the stabilizer of the element Σ_+ in $\mathfrak{psu}(2, 2|4)$:

$$[\Sigma_+, \mathcal{M}] = 0, \quad \mathcal{M} \in \mathcal{J}.$$

Obviously, \mathcal{J} coincides with the red-blue submatrix of \mathcal{M} in Figure 1. Thus, for P_+ being finite⁵ we would obtain the following vector space decomposition of \mathcal{J}

$$\mathcal{J} = \mathfrak{psu}(2|2) \oplus \mathfrak{psu}(2|2) \oplus \Sigma_+ \oplus \Sigma_-.$$

The rank of the latter subalgebra is six and it coincides with that of $\mathfrak{psu}(2, 2|4)$. In the case of infinite P_+ the last generator decouples.

Now we are ready to determine the general form of the central term in eq.(3.9). Denote by $\mathcal{J}_{\text{even}} \subset \mathcal{J}$ the even (bosonic) subalgebra of \mathcal{J} . It is represented by the red and blue diagonal blocks in Figure 1. Let G_{even} be the corresponding group. The adjoint action of G_{even} preserves the \mathbb{Z}_2 -grading of \mathcal{J} . Obviously, if we perform the transformation

$$Q \rightarrow gQg^{-1}, \quad \mathcal{M} \rightarrow g^{-1}\mathcal{M}g$$

⁵As a side remark, we note that for P_+ finite the subalgebra which leaves invariant both H and P_+ coincides with the even (bosonic) subalgebra $\mathcal{J}_{\text{even}}$ of \mathcal{J} . According to eqs.(3.6) and (3.7), this subalgebra arises as the simultaneous solution the two equations, $[\Sigma_+, \mathcal{M}] = 0$ and $[\Sigma_-, \mathcal{M}] = 0$, and it is represented by the red and blue diagonal blocks in Figure 1.

with an element $g \in G_{\text{even}}$ the charge $Q_{\mathcal{M}}$ remains invariant. This transformation leaves the l.h.s of the bracket (3.9) invariant. Thus, the central term must satisfy the following invariance condition:

$$C(g\mathcal{M}_1g^{-1}, g\mathcal{M}_2g^{-1}) = C(\mathcal{M}_1, \mathcal{M}_2). \quad (3.10)$$

It is not difficult to find a general expression for a bilinear graded skew-symmetric form on \mathcal{J} which satisfies this condition. It is given by

$$C(\mathcal{M}_1, \mathcal{M}_2) = \text{str} \left((\varrho\mathcal{M}_1\varrho\mathcal{M}_2^t + (-1)^{\pi(\mathcal{M}_1)\pi(\mathcal{M}_2)}\varrho\mathcal{M}_2\varrho\mathcal{M}_1^t) \Delta \right). \quad (3.11)$$

Here

$$\Delta = -\frac{1}{2} \begin{pmatrix} c_3 \mathbb{I}_2 & 0 & 0 & 0 \\ 0 & c_1 \mathbb{I}_2 & 0 & 0 \\ 0 & 0 & c_4 \mathbb{I}_2 & 0 \\ 0 & 0 & 0 & c_2 \mathbb{I}_2 \end{pmatrix}, \quad (3.12)$$

where \mathbb{I}_2 is the two-dimensional identity matrix and

$$\varrho = \begin{pmatrix} \sigma & 0 & 0 & 0 \\ 0 & \sigma & 0 & 0 \\ 0 & 0 & \sigma & 0 \\ 0 & 0 & 0 & \sigma \end{pmatrix}, \quad \sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (3.13)$$

Condition (3.10) follows from the form of the matrix Δ and the equation

$$\mathcal{J}_{\text{even}}^t \varrho + \varrho \mathcal{J}_{\text{even}} = 0. \quad (3.14)$$

The coefficients c_i , $i = 1, \dots, 4$ can depend on the physical fields and they are central w.r.t. the action of \mathcal{J} :

$$\{c_i, Q_{\mathcal{M}}\} = 0, \quad \mathcal{M} \in \mathcal{J}.$$

By using eq.(3.11) one can check that the cocycle condition for $C(\mathcal{M}_1, \mathcal{M}_2)$ is trivially satisfied. In accordance with our assumptions, $C(\mathcal{M}_1, \mathcal{M}_2)$ does not vanish only if both \mathcal{M}_1 and \mathcal{M}_2 are odd.

Taking into account that \mathcal{J} contains two identical subalgebras $\mathfrak{psu}(2|2)$ we can put $c_1 = c_3$ and $c_2 = c_4$. Thus, general symmetry arguments fix the form of the central extension up to two central functions c_1 and c_2 . Since we consider the algebra $\mathfrak{psu}(2, 2)$, which is the real form of $\mathfrak{psl}(2|2)$, the conjugation rule implies that $c_1 = -c_2^*$. In the next section we compute the Poisson brackets of the Noether charges and determine the explicit form of $c \equiv c_1$.

3.3 Explicit basis

In what follows we find it convenient to pick up a basis in the space of the Noether charges $Q_{\mathcal{M}}$ with $\mathcal{M} \in \mathcal{J}$ and write down the bracket (3.9) for the corresponding

basis elements. Since \mathcal{J} contains two identical $\mathfrak{psu}(2, 2)$ subalgebras it is enough to analyze the Poisson brackets corresponding to one of them. For definiteness, we concentrate our attention on the subalgebra $\mathfrak{psu}(2, 2)_R$ which is represented in Figure 1 by blue (right) blocks. For this subalgebra we select a basis in which the fermionic (dynamical) generators are given by

$$\begin{aligned} Q_1^1 &= \frac{1}{2} \text{str } Q \sigma^+ \otimes (\Gamma_{14} - \Gamma_{23} + \mathcal{P}_-) , & Q_2^1 &= \text{str } Q \sigma^+ \otimes \Gamma_{13} , \\ Q_2^2 &= -\frac{1}{2} \text{str } Q \sigma^+ \otimes (\Gamma_{14} - \Gamma_{23} - \mathcal{P}_-) , & Q_1^2 &= -\text{str } Q \sigma^+ \otimes \Gamma_{42} \end{aligned} \quad (3.15)$$

and their conjugate charges \bar{Q}_a^α are

$$\begin{aligned} \bar{Q}_1^1 &= -\frac{1}{2} \text{str } Q \sigma^- \otimes (\Gamma_{14} - \Gamma_{23} + \mathcal{P}_-) , & \bar{Q}_2^1 &= -\text{str } Q \sigma^- \otimes \Gamma_{13} , \\ \bar{Q}_2^2 &= \frac{1}{2} \text{str } Q \sigma^- \otimes (\Gamma_{14} - \Gamma_{23} - \mathcal{P}_-) , & \bar{Q}_1^2 &= \text{str } Q \sigma^- \otimes \Gamma_{42} . \end{aligned} \quad (3.16)$$

On the other hand, the bosonic (kinematical) charges are defined as

$$\begin{aligned} L_1^1 &= \frac{i}{2} \text{str } Q P_2^+ \otimes (\Gamma_{14} - \Gamma_{23}) , & L_2^2 &= -L_1^1 , \\ L_2^1 &= -i \text{str } Q P_2^+ \otimes \Gamma_{13} , & L_1^2 &= -i \text{str } Q P_2^+ \otimes \Gamma_{24} , \end{aligned} \quad (3.17)$$

and

$$\begin{aligned} R_1^1 &= \frac{i}{2} \text{str } Q P_2^- \otimes (\Gamma_{14} - \Gamma_{23}) , & R_2^2 &= -R_1^1 , \\ R_2^1 &= i \text{str } Q P_2^- \otimes \Gamma_{13} , & R_1^2 &= i \text{str } Q P_2^- \otimes \Gamma_{24} , \end{aligned} \quad (3.18)$$

We refer the reader to the appendix 6.1. for notations used here.

Rewriting the bracket (3.9) in this basis we obtain the following relations

$$\begin{aligned} \{R_b^a, J^c\} &= \delta_b^c J^a - \frac{1}{2} \delta_b^a J^c , & \{L_\beta^\alpha, J^\gamma\} &= \delta_\beta^\gamma J^\alpha - \frac{1}{2} \delta_\beta^\alpha J^\gamma , \\ \{Q_a^\alpha, \bar{Q}_\beta^b\} &= \delta_a^b L_\beta^\alpha + \delta_\beta^a R_a^b + \frac{1}{2} \delta_a^b \delta_\beta^\alpha H , \\ \{Q_a^\alpha, Q_b^\beta\} &= \epsilon^{\alpha\beta} \epsilon_{ab} c , & \{\bar{Q}_\alpha^a, \bar{Q}_\beta^b\} &= \epsilon^{ab} \epsilon_{\alpha\beta} c^* . \end{aligned} \quad (3.19)$$

Here in the first line we have indicated how the indices a and γ of any Lie algebra generator transform under the action of the bosonic subalgebras generated by R_b^a and L_β^α . In the next section we are going to derive the so far undetermined central function c in terms of physical variables of string theory.

4. Deriving the central charges

Given the complex structure of the supersymmetry generators in the light-cone gauge as well as the corresponding Poisson structure of the theory, the direct computation

of the classical and quantum supersymmetry algebra does not seem to be feasible. Hence, simplifying perturbative methods need to be applied. The perturbative expansion of the supersymmetry generators in powers of $\zeta = \frac{2\pi}{\sqrt{\lambda}}$ or, equivalently, in the number of fields defines a particular expansion scheme. This expansion, however, does not allow one to determine the exact form of the central charges because they are also expected to be non-trivial functions of ζ . To overcome this difficulty in this section we describe a “hybrid” expansion scheme which can be used to determine the exact form of the central charges. To be precise we determine only the part of the central charges which is independent of fermionic fields. We find that this part depends solely on the piece of the level-matching constraint which involves the bosonic fields. Since the central charges must vanish on the level-matching constraint surface, the exact form of the central charges is, therefore, unambiguously fixed by its bosonic part.

More precisely, a dynamical supersymmetry generator has the following generic structure

$$Q_{\mathcal{M}} = \int d\sigma e^{i\alpha x_-} \Omega(x, p, \chi; \zeta). \quad (4.1)$$

Depending on the supercharge, the parameter α in the exponent of (4.1) can take two values $\alpha = \pm \frac{1}{2}$. Here the function $\Omega(x, p, \chi; \zeta)$ is a *local* function of transverse bosonic fields and fermionic variables. It depends on ζ , and can be expanded, quite analogous to the Hamiltonian, in power series

$$\Omega(x, p, \chi; \zeta) = \Omega_2(x, p, \chi) + \zeta \Omega_4(x, p, \chi) + \cdots$$

Here $\Omega_2(x, p, \chi)$ is quadratic in fields, $\Omega_4(x, p, \chi)$ is quartic and so on. Clearly, every term in this series also admits a finite expansion in number of fermions. In the usual perturbative expansion we would also have to expand the non-local “vertex” $e^{i\alpha x_-}$ in powers of ζ because $x'_- \sim -\zeta p x' + \cdots$. In the hybrid expansion we do not expand $e^{i\alpha x_-}$ but rather treat it as a rigid object.

The complete expression for a supercharge is rather cumbersome. However, we see that the supercharges and their algebra can be studied perturbatively: first by expanding up to the given order in ζ and then by truncating the resulting polynomial up to the given number of fermionic variables. Then, as was discussed above the exact form of the central charges is completely fixed by their parts which depend only on bosons. Thus, to determine these charges it is sufficient to consider the terms in $Q_{\mathcal{M}}$ which are linear in fermions, and compute their Poisson brackets (or anticommutators in quantum theory) keeping only terms independent of fermions. This is, however, a complicated problem because the Poisson brackets of fermions appearing in (3.1) have a highly non-trivial dependence on bosons, see [11] for details. It was shown in [11] that to have the canonical Poisson brackets one should perform

a field redefinition which can be determined up to any given order in ζ . Taking into account the field redefinition, integrating by parts if necessary, and using the relation $x'_- \sim -\zeta p x' + \dots$, one can cast any supercharge (4.1) to the following symbolic form

$$Q_{\mathcal{M}} = \int d\sigma e^{i\alpha x_-} \chi \cdot (\Upsilon_1(x, p) + \zeta \Upsilon_3(x, p) + \dots) + \mathcal{O}(\chi^3), \quad (4.2)$$

where Υ_1 and Υ_3 are linear and cubic in bosonic fields, respectively. The explicit form of the supercharges expanded up to the order ζ can be found in the Appendix.

It is clear now that the bosonic part of the Poisson bracket of two supercharges is of the form

$$\begin{aligned} \{Q_1, Q_2\} \sim \int_{-\infty}^{\infty} d\sigma e^{i(\alpha_1 + \alpha_2)x_-} & \left(\Upsilon_1^{(1)}(x, p) \Upsilon_1^{(2)}(x, p) \right. \\ & \left. + \zeta (\Upsilon_1^{(1)}(x, p) \Upsilon_3^{(2)}(x, p) + \Upsilon_3^{(1)}(x, p) \Upsilon_1^{(2)}(x, p)) + \dots \right), \end{aligned} \quad (4.3)$$

where $Q_{1,2} \equiv Q_{\mathcal{M}_{1,2}}$. Computing the product $\Upsilon_1^{(1)}(x, p) \Upsilon_1^{(2)}(x, p)$ in the case $\alpha_1 = \alpha_2 = \pm 1/2$, we find that it is given by

$$\Upsilon_1^{(1)}(x, p) \Upsilon_1^{(2)}(x, p) \sim \frac{1}{\zeta} x'_- + \frac{d}{d\sigma} f(x, p), \quad (4.4)$$

where $f(x, p)$ is a local function of transverse coordinates and momenta. The first term in (4.4) nicely combines with $e^{\pm i x_-}$ to give $\frac{d}{d\sigma} e^{\pm i x_-}$, and integrating this expression over σ , we obtain the sought for central charges

$$\int_{-\infty}^{\infty} d\sigma \frac{d}{d\sigma} e^{\pm i x_-} = e^{\pm i x_-(\infty)} - e^{\pm i x_-(-\infty)} = e^{\pm i x_-(\infty)} (e^{\pm i p_{\text{ws}}} - 1), \quad (4.5)$$

where we take into account that $x_-(\infty) - x_-(-\infty) = p_{\text{ws}}$.

Making use of the particular basis described in the previous section and imposing the boundary condition $x_-(-\infty) = 0$, we identify the exact expression for the central function c in eqs.(3.19):

$$c = \frac{1}{2\zeta} (e^{i p_{\text{ws}}} - 1). \quad (4.6)$$

The algebra (3.19) with the central charges of the form (4.6) perfectly agrees with the one considered in [5] in the field theory context.

It is worth noting that there is another natural choice of the boundary condition for the light-cone coordinate x_- :

$$x_-(+\infty) = -x_-(-\infty) = \frac{p_{\text{ws}}}{2}.$$

This is the symmetric condition which treats both boundaries on the equal footing, and leads to a purely imaginary central charge

$$c = \frac{i}{\zeta} \sin\left(\frac{p_{\text{ws}}}{2}\right). \quad (4.7)$$

It is obvious from (4.5) that different boundary conditions for x_- lead to central charges which differ from each other by a phase multiplication. This freedom in the choice of the central charge follows from the obvious $U(1)$ automorphism of the algebra (3.19): one can multiply all supercharges Q_a^α by any phase which in general may depend on all the central charges.

Since we already obtained the expected central charges, the contribution of all the other terms in (4.3) should vanish. Indeed, the second term in (4.4) contributes to the order ζ in the expansion as can be easily seen integrating by parts and using the relation $x'_- \sim -\zeta p x' + \dots$. Taking into account the additional contribution to the terms of order ζ in (4.3), we have checked that the total contribution is given by a σ -derivative of a local function of x and p , and, therefore, only contributes to terms of order ζ^2 .

We have verified up to the quartic order in fields that the Poisson bracket of supercharges with $\alpha_1 = -\alpha_2$ gives the Hamiltonian and the kinematic generators in complete agreement with the centrally extended $\mathfrak{su}(2|2)$ algebra (3.19).

The next step is to show that the Hamiltonian commutes with all dynamical supercharges. As was already mentioned, this can be done order by order in perturbation theory in powers of the inverse string tension ζ and in number of fermionic variables. We have demonstrated that up to the first non-trivial order ζ the supercharge Q truncated to the terms linear in fermions indeed commutes with H . To do that we need to keep in H all quadratic and quartic bosonic terms, and quadratic and quartic terms which are quadratic in fermions, see the Appendix for details.

The computation we described above was purely classical, and one may want to know if quantizing the model could lead to some kind of an anomaly in the symmetry algebra. We have computed the symmetry algebra in the plane-wave limit where one keeps only quadratic terms in all the symmetry generators, and shown that all potentially divergent terms cancel out and no quantum anomaly arises. As a result, one gets again the same centrally extended $\mathfrak{su}(2|2)$ algebra (3.19) with the central charges $1/\zeta(e^{\pm i p_{\text{ws}}} - 1)$ replaced by their low-momentum approximations $\mp i \int_{-\infty}^{\infty} d\sigma \left(p_M x'_M - \frac{i}{2} \text{str}(\Sigma_+ \chi \chi') \right)$.

Thus, we have shown that in the infinite P_+ limit and for physical fields chosen to rapidly decrease at infinity the corresponding string model enjoys the symmetry which coincides with two copies of the centrally-extended $\mathfrak{su}(2|2)$ algebra (3.19) sharing the same Hamiltonian.

5. Concluding remarks

The main focus of this paper has been on the analysis of the off-shell string symmetry algebra in the limit of *infinite* light-cone momentum P_+ . Relaxing the level-matching condition brings only one modification in this case: namely, the algebra $\mathfrak{psu}(2|2) \oplus \mathfrak{psu}(2|2)$ undergoes extension by a new central charge proportional to the level-matching condition.

The physically more relevant situation, however, corresponds to the case of a finite light-cone momentum. For P_+ finite the zero mode of the conjugate field x_- has to be taken into account. Also, since the length of the string is finite, transverse fields do not have to vanish at the string end points. So the question arises what is the symmetry algebra in this case?

Recall that relaxing the level-matching condition for finite P_+ means

$$x_-(r) - x_-(-r) = p_{\text{ws}}, \quad -r \leq \sigma \leq r, \quad r = \frac{\pi P_+}{\sqrt{\lambda}},$$

which implies that the Poisson bracket of the dynamical supercharge Q , eq.(4.1), with the level-matching generator is

$$\frac{1}{\zeta} \{p_{\text{ws}}, Q\} = \int_{-r}^r d\sigma \partial_\sigma Q = \Omega(r) e^{i\alpha x_-(-r)} (e^{i\alpha p_{\text{ws}}} - 1) \neq 0.$$

Hence this Poisson bracket does not vanish, since $\Omega(r) = \Omega(-r)$ is non-zero in the finite P_+ case. Similarly the Poisson bracket of the same supercharge with the Hamiltonian will be non-vanishing. Thus, we see that the off-shell extension of the theory does not allow one to maintain the $\mathfrak{psu}(2, 2|4)$ symmetry algebra for a string of finite length.

It should be further noted that an off-shell theory is not uniquely defined. Indeed, one can use the level-matching generator to modify the Hamiltonian

$$H \rightarrow H + c_n p_{\text{ws}}^n,$$

where the coefficients c_n might depend on physical fields. On the states satisfying the condition of level-matching the new Hamiltonian reduces to the original one. The absence of the standard $\mathfrak{psu}(2, 2|4)$ symmetry in an off-shell theory does not a priori preclude the existence of new hidden symmetries of the off-shell Hamiltonian. Their discovery would provide a substantial step in understanding the string dynamics for the physically relevant situation of the finite light-cone momentum.

In the hybrid expansion used in our paper, the crucial role in deriving the non-linear central charges, was played by the “vertex” $e^{i\alpha x^-}$. The question is what is

the physical meaning of this object? To see this, consider the quantum theory. The variable $x_-(s)$ contains a zero mode ¹ \hat{x}_- which is conjugate to the operator \hat{P}_+

$$[\hat{P}_+, \hat{x}_-] = -i.$$

Thus, if we consider a state $|P_+\rangle$ with a definite value of $\hat{P}_+|P_+\rangle = P_+|P_+\rangle$ then a state $e^{i\alpha\hat{x}_-}|P_+\rangle$ carries a new value of P_+ :

$$\hat{P}_+e^{i\alpha\hat{x}_-}|P_+\rangle = (\alpha + P_+)e^{i\alpha\hat{x}_-}|P_+\rangle$$

Since in the light-cone approach P_+ is naturally identified with the length of the string, it is appropriate to call $e^{i\alpha\hat{x}_-}$ the *length-changing operator*. The Hilbert space of the corresponding theory is necessarily a direct sum: $\mathcal{H} = \sum_{P_+} \mathcal{H}_{P_+}$ of the spaces \mathcal{H}_{P_+} corresponding to an individual eigenvalue of the operator \hat{P}_+ .

This brief discussion of the light-cone string theory for finite P_+ clearly demonstrates that the latter carries many subtleties with respect to the infinite P_+ limit, which for sure require further investigation.

Acknowledgements

We wish to thank Niklas Beisert for valuable discussions. The work of G. A. was supported in part by the RFBI grant N05-01-00758, by NWO grant 047017015 and by the INTAS contract 03-51-6346. The work of G. A. and S. F. was supported in part by the EU-RTN network *Constituents, Fundamental Forces and Symmetries of the Universe* (MRTN-CT-2004-005104). The work of J. P. is supported by the Volkswagen Foundation. He also thanks the Albert-Einstein-Institute for hospitality. The work of M. Z. was supported in part by the grant *Superstring Theory* (MRTN-CT-2004-512194). M. Z. would like to thank the Spinoza Insitute for the hospitality during the last phase of this project.

6. Appendix

6.1 The gauge-fixed Hamiltonian

The Hamiltonian for physical excitations arising in the light-cone gauge was found in [11]. The gauge choice made in [11] is however not exactly the same as eq.(2.2) adopted here. Also the theory in [11] is defined on the standard interval for σ : $-\pi \leq \sigma \leq \pi$. In order to make a connection with the results by [11] we choose the variable p_+ there to be equal to $p_+ = 2P_+$, where P_+ is identified with the

¹This is an integration constant arising upon integrating equation.(3.4).

total momentum in (2.3). To justify our choice we note that with $p_+ = 2P_+$ the gauge-fixed action of [11] can be schematically represented in the form

$$S = P_+ \int_{-\pi}^{\pi} \frac{d\sigma d\tau}{2\pi} \left(p_M \dot{x}_M + \chi^\dagger \dot{\chi} - \mathcal{H} \right), \quad (6.1)$$

where \mathcal{H} is the P_+ -independent Hamiltonian density, (x_M, p_M) with $M = 1, \dots, 8$ are transverse coordinates and their conjugate momenta, and χ encodes the fermionic variables. Since we are interested in the limit of infinite P_+ , it is appropriate to make a rescaling $\sigma \rightarrow \frac{\sqrt{\lambda}}{P_+} \sigma$. Upon this rescaling the action (6.1) turns precisely into eq.(2.4) of the present paper with $r = \frac{\pi}{\sqrt{\lambda}} P_+$. As was discussed in section 2, we further supplement this rescaling of σ with rescalings of physical variables

$$(x_M, p_M, \chi) \rightarrow \sqrt{\zeta} (x_M, p_M, \chi). \quad (6.2)$$

This is necessary in order to ensure to have canonical Poisson brackets for physical fields.

Upon these redefinitions the Hamiltonian found in [11] turns into the one given by eq.(2.6). Explicitly, the quadratic piece of the Hamiltonian density in eq.(2.6) has the form

$$\mathcal{H}_2 = \frac{1}{2} p_M^2 + \frac{1}{2} x_M^2 + \frac{1}{2} x_M'^2 + \frac{1}{2} \text{str}(\Sigma_+ \chi \tilde{K} \chi'^t K) + \frac{1}{2} \text{str} \chi^2, \quad (6.3)$$

while the quartic one is [11]

$$\begin{aligned} \mathcal{H}_4 = \frac{1}{4} & \left[p_y^2 z^2 - p_z^2 y^2 + (y'^2 z^2 - z'^2 y^2) + 2(z'^2 z^2 - y'^2 y^2) + \right. \\ & + \text{str} \left((z^2 - y^2) \chi' \chi' + \frac{1}{2} [\Sigma'(x), \Sigma(x)] (\chi \chi' - \chi' \chi) - 2 \Sigma(x) \chi' \Sigma(x) \chi' \right) \\ & \left. + \frac{i}{4} \text{str} \left([\Sigma(x), \Sigma(p)]' (\tilde{K} \chi^t K \chi - \chi \tilde{K} \chi^t K) \right) \right] + \mathcal{O}(\chi^4), \end{aligned} \quad (6.4)$$

where by $\mathcal{O}(\chi^4)$ we encode all the terms which are quartic in fermions (stated in [11]). The transverse bosonic fields we have denoted as $x_M = (z_a, y_s)$ with z_a ($a = 1, 2, 3, 4$) accounting for the transverse AdS_5 and y_s ($s = 1, 2, 3, 4$) for the S^5 degrees of freedom. Prime denotes ∂_σ and in the fermionic sector we have introduced the following notation

$$K = \begin{pmatrix} K_4 & 0 \\ 0 & K_4 \end{pmatrix}, \quad \tilde{K} = \begin{pmatrix} K_4 & 0 \\ 0 & -K_4 \end{pmatrix}.$$

with the matrix K_4 satisfying $K_4^2 = -\mathbb{I}$ given by

$$K_4 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

We also use the notation $\Sigma(x) = \Sigma_M x_M$ and $\Sigma(p) = \Sigma_M p_M$. The 8×8 matrices Σ_M have the following structure

$$\Sigma_M = \left\{ \begin{pmatrix} \gamma_a & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & i\gamma_s \end{pmatrix} \right\} \quad (6.5)$$

and are written in terms of the four Dirac matrices γ_i . We work with the basis defined in appendix A of [11]. For the definition of the matrices Σ_{\pm} see eq.(3.3).

The fermions enter in the above through the κ -gauge fixed 8×8 matrix χ (compare (A.6) and (A.9) of [11])

$$\chi = \begin{pmatrix} 0 & \mathcal{P}_+ \eta + \mathcal{P}_- \theta^\dagger \\ -\mathcal{P}_- \eta^\dagger + \mathcal{P}_+ \theta & 0 \end{pmatrix}, \quad \mathcal{P}_+ = \begin{pmatrix} \mathbb{I}_2 & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathcal{P}_- = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{I}_2 \end{pmatrix} \quad (6.6)$$

where

$$\eta = \sum_{i=1}^4 \tilde{\eta}_i \Gamma_i, \quad \theta = \sum_{i=1}^4 \tilde{\theta}_i \Gamma_i, \quad (6.7)$$

with the Dirac matrices Γ_i in the complex basis defined in [11], explicitly

$$\Gamma_1 = \frac{1}{2}(\gamma_2 - i\gamma_1) = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \Gamma_2 = \frac{1}{2}(\gamma_4 - i\gamma_3) = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}$$

and $\Gamma_4 = (\Gamma_1)^\dagger$, $\Gamma_3 = (\Gamma_2)^\dagger$. Moreover we define the standard double index Dirac matrices by $\Gamma_{ab} := \frac{1}{2}[\Gamma_a, \Gamma_b]$. We also define two-dimensional projectors

$$P_2^+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}; \quad P_2^- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \quad (6.8)$$

6.2 Symmetry generators

To describe the symmetry generators Q and the gauge-fixed Hamiltonian we have to introduce a proper parametrization of the coset space $\frac{\text{PSU}(2,2|4)}{\text{SO}(4,1) \times \text{SO}(5)}$. Following [11] we chose the coset representative in the form

$$g(\chi, x, t, \phi) = \Lambda(t, \phi) g(\chi) g(x).$$

Here $x_M = (z_a, y_i)$ is a short-hand notation for the transverse bosonic fields and χ denotes the 16 physical fermions which are left upon fixing the κ -symmetry. The matrix $\Lambda(t, \phi)$ was defined in (3.2). The element $g(x)$ is the 8×8 matrix which has the following structure in terms of 4×4 blocks related to the AdS and to the sphere parts respectively

$$g(x) = \begin{pmatrix} \frac{1}{\sqrt{1-\zeta \frac{z^2}{4}}} \left(1 + \frac{\sqrt{\zeta}}{2} z_a \gamma_a \right) & 0 \\ 0 & \frac{1}{\sqrt{1+\zeta \frac{y^2}{4}}} \left(1 + \frac{i\sqrt{\zeta}}{2} y_i \gamma_i \right) \end{pmatrix}.$$

Finally the fermionic coset element reads [11]

$$g(\chi) = \sqrt{\zeta} \chi + \sqrt{1 + \zeta \chi^2}.$$

The 8×8 supermatrix Q of (3.1) is then defined by [11]

$$Q = \int_{-r}^r d\sigma \Lambda U \Lambda^{-1}, \quad (6.9)$$

where

$$U = g(\chi)g(x) \left(\pi + \frac{i}{2}g(x)\tilde{K}F_\sigma^t K g(x)^{-1} \right) g(x)^{-1}g(\chi)^{-1}. \quad (6.10)$$

Here K and \tilde{K} have been defined above. We also have

$$F_\sigma = \sqrt{\zeta} (\sqrt{1 + \zeta \chi^2} \partial_\sigma \chi - \chi \partial_\sigma \sqrt{1 + \zeta \chi^2})$$

and π is defined by

$$\pi = \frac{i}{4}\pi_{+\Sigma_+} + \frac{i}{4}\pi_{-\Sigma_-} + \frac{1}{2}\pi_M \Sigma_M,$$

where

$$\begin{aligned} \pi_+ &= \frac{1}{G_+}(2 + G_- \pi_-), & \pi_- &= -\frac{G_+(\pi_M^2 + \mathcal{A}^2)}{(1 + \sqrt{1 - G_+ G_-}(\pi_M^2 + \mathcal{A}^2))}, \\ \pi_a &= \sqrt{\zeta} p_a (1 - \zeta \frac{z^2}{4}), & \pi_s &= \sqrt{\zeta} p_s (1 + \zeta \frac{y^2}{4}). \end{aligned}$$

Finally \mathcal{A}^2 and G_\pm are given by

$$\mathcal{A}^2 = -x_-'^2 G_+ G_- + \frac{\zeta z_a'^2}{(1 - \zeta \frac{z^2}{4})^2} + \mathcal{O}(\chi^2), \quad G_\pm = \frac{1}{2} \left(\frac{1 + \zeta \frac{z^2}{4}}{1 - \zeta \frac{z^2}{4}} \pm \frac{1 - \zeta \frac{y^2}{4}}{1 + \zeta \frac{y^2}{4}} \right).$$

6.3 Covariant notation

As was done in [11] we shall make use of complex fields. However, here we will denote them in a covariant notation with upper and lower indices reflecting their charges under the four transverse $U(1)$ subgroups involved. The bosonic fields we denote by

$$\begin{aligned} Z_1 &= z_2 + iz_1; & Z_2 &= z_4 + iz_3; & Z^2 &= (Z_2)^\dagger; & Z^1 &= (Z_1)^\dagger; \\ Y_1 &= y_2 + iy_1; & Y_2 &= y_4 + iy_3; & Y^2 &= (Y_2)^\dagger; & Y^1 &= (Y_1)^\dagger; \\ P_1^Z &= \frac{1}{2}(p_2^z + ip_1^z); & P_2^Z &= \frac{1}{2}(p_4^z + ip_3^z); & (P^Z)^1 &= (P_1^Z)^\dagger; & (P^Z)^2 &= (P_2^Z)^\dagger; \\ P_1^Y &= \frac{1}{2}(p_2^y + ip_1^y); & P_2^Y &= \frac{1}{2}(p_4^y + ip_3^y); & (P^Y)^1 &= (P_1^Y)^\dagger; & (P^Y)^2 &= (P_2^Y)^\dagger. \end{aligned}$$

with the quantum commutation relations ($\alpha, \beta = 1, 2$ and $a, b = 1, 2$)

$$\begin{aligned} [P_\alpha^Z(\sigma), Z^\beta(\sigma')] &= -i \delta_\alpha^\beta \delta(\sigma - \sigma') & [(P^Z)^\alpha(\sigma), Z_\beta(\sigma')] &= -i \delta_\beta^\alpha \delta(\sigma - \sigma') \\ [P_a^Y(\sigma), Y^b(\sigma')] &= -i \delta_a^b \delta(\sigma - \sigma') & [(P^Y)^a(\sigma), Y_b(\sigma')] &= -i \delta_b^a \delta(\sigma - \sigma'), \end{aligned} \quad (6.11)$$

analogue expressions apply at the classical level for the Poisson brackets.

We also introduce upper and lower indices for the fermionic fields defined in (6.7) by denoting

$$\begin{aligned}
\tilde{\theta}_1 &= \theta_1 & \tilde{\theta}_2 &= \theta_2 & \tilde{\theta}_3 &= \theta^2 & \tilde{\theta}_4 &= \theta^1 \\
\tilde{\theta}_1^\dagger &= \theta^{\dagger 1} & \tilde{\theta}_2^\dagger &= \theta^{\dagger 2} & \tilde{\theta}_3^\dagger &= \theta_2^\dagger & \tilde{\theta}_4^\dagger &= \theta_1^\dagger \\
\tilde{\eta}_1 &= \eta_1 & \tilde{\eta}_2 &= \eta_2 & \tilde{\eta}_3 &= \eta^2 & \tilde{\eta}_4 &= \eta^1 \\
\tilde{\eta}_1^\dagger &= \eta^{\dagger 1} & \tilde{\eta}_2^\dagger &= \eta^{\dagger 2} & \tilde{\eta}_3^\dagger &= \eta_2^\dagger & \tilde{\eta}_4^\dagger &= \eta_1^\dagger,
\end{aligned} \tag{6.12}$$

leading to the covariant anti-commutation relations

$$\begin{aligned}
\{\theta_\alpha(\sigma), \theta^{\dagger\beta}(\sigma')\} &= \delta_\alpha^\beta \delta(\sigma - \sigma') & \{\theta^\alpha(\sigma), \theta_\beta^\dagger(\sigma')\} &= \delta_\beta^\alpha \delta(\sigma - \sigma') \\
\{\eta_a(\sigma), \eta^{\dagger b}(\sigma')\} &= \delta_a^b \delta(\sigma - \sigma') & \{\eta^a(\sigma), \eta_b^\dagger(\sigma')\} &= \delta_b^a \delta(\sigma - \sigma').
\end{aligned} \tag{6.13}$$

It is useful to note the charges carried by the fields of the four $U(1)$ subgroups involved. For this consider the combinations $S_\pm = S_1 \pm S_2$ and $J_\pm = J_1 \pm J_2$. Then the $\mathfrak{su}(2|2)_R$ right (blue) generators carry J_- and S_- charges whereas the $\mathfrak{su}(2|2)_L$ left (red) generators are charged under S_- and J_- . The following tables exemplify this:

	S_+	S_-	J_+	J_-
$Z_1, (P^Z)_1, \bar{Z}_1, (\bar{P}^Z)_1,$	1	1	0	0
$Z_2, (P^Z)_2, \bar{Z}^2, (\bar{P}^Z)^2,$	1	-1	0	0
$Z^2, (P^Z)^2, \bar{Z}_2, (\bar{P}^Z)_2,$	-1	1	0	0
$Z^1, (P^Z)^1, \bar{Z}^1, (\bar{P}^Z)^1,$	-1	-1	0	0

	S_+	S_-	J_+	J_-
$Y_1, (P^Y)_1, \bar{Y}_1, (\bar{P}^Y)_1,$	0	0	1	1
$Y_2, (P^Y)_2, \bar{Y}^2, (\bar{P}^Y)^2,$	0	0	1	-1
$Y^2, (P^Y)^2, \bar{Y}_2, (\bar{P}^Y)_2,$	0	0	-1	1
$Y^1, (P^Y)^1, \bar{Y}^1, (\bar{P}^Y)^1,$	0	0	-1	-1

	S_+	S_-	J_+	J_-
$\theta_1, \theta_1^\dagger, \bar{\theta}_1, \bar{\theta}_1^\dagger,$	0	1	1	0
$\theta_2, \theta_2^\dagger, \bar{\theta}^2, \bar{\theta}^{\dagger 2},$	0	-1	1	0
$\theta^2, \theta^{\dagger 2}, \bar{\theta}_2, \bar{\theta}_2^\dagger,$	0	1	-1	0
$\theta^1, \theta^{\dagger 1}, \bar{\theta}^1, \bar{\theta}^{\dagger 1},$	0	-1	-1	0

	S_+	S_-	J_+	J_-
$\eta_1, \eta_1^\dagger, \bar{\eta}_1, \bar{\eta}_1^\dagger,$	1	0	0	1
$\eta_2, \eta_2^\dagger, \bar{\eta}^2, \bar{\eta}^{\dagger 2},$	1	0	0	-1
$\eta^2, \eta^{\dagger 2}, \bar{\eta}_2, \bar{\eta}_2^\dagger,$	-1	0	0	1
$\eta^1, \eta^{\dagger 1}, \bar{\eta}^1, \bar{\eta}^{\dagger 1},$	-1	0	0	-1

Hence a lower (upper) index on $Z, Y, P^Z, P^Y, \theta, \eta, \theta^\dagger$ and η^\dagger denotes a charge of 1 (-1) with respect to $(S_+ + J_+)$. In the above tables we have also introduced barred coordinates defined with a flipped “2” index as

$$\bar{A}_1 = A_1, \quad \bar{A}^1 = A^1, \quad \bar{A}_2 = A^2, \quad \bar{A}^2 = A_2, \quad \text{with } A \in \{Z, Y, P^Z, P^Y, \theta, \eta, \theta^\dagger, \eta^\dagger\},$$

which are the natural objects for the $\mathfrak{su}(2|2)_L$ left (red) generators as we shall see shortly. For the barred coordinates the index position now denotes the charge with respect to $(S_- + J_-)$. Clearly the commutation relations keep their canonical form, cf. (6.11) and (6.13), in the barred coordinates.

6.4 The explicit form of the $\mathfrak{su}(2|2)_R$ generators

Using the basis of fermionic (dynamical) generators of the right (blue) $\mathfrak{su}(2|2)_R$ algebra given in (3.15) and (3.16) along with the concrete expression for Q in (6.9) one finds the leading quadratic order expressions for the supercharges

$$Q^\alpha_a = -\frac{1}{2} \int d\sigma e^{-\frac{i}{2}x-} \left[i\theta^\alpha (2P^Y + iY)_a + (2P^Z - iZ)^\alpha \eta_a^\dagger - \theta^{\dagger\alpha} Y'_a - iZ'^\alpha \eta_a \right. \\ \left. + \epsilon^{\alpha\beta} \epsilon_{ab} \left(i\theta_\beta (2P^Y + iY)^b + (2P^Z - iZ)_\beta \eta^{\dagger b} - \theta_\beta^\dagger Y'^b - iZ'_\beta \eta^b \right) \right] \quad (6.14)$$

$$\bar{Q}_\alpha^a = \frac{1}{2} \int d\sigma e^{\frac{i}{2}x-} \left[i\theta_\alpha^\dagger (2P^Y - iY)^a - (2P^Z + iZ)_\alpha \eta^a + \theta_\alpha Y'^a - iZ'_\alpha \eta^{\dagger a} \right. \\ \left. + \epsilon_{\alpha\beta} \epsilon^{ab} \left(i\theta^{\dagger\beta} (2P^Y - iY)_b - (2P^Z + iZ)^\beta \eta_b + \theta^\beta Y'_b - iZ'^\beta \eta_b^\dagger \right) \right] = (Q^\alpha_a)^\dagger. \quad (6.15)$$

Moreover the $\mathfrak{su}(2)$ generators R^α_β and L^a_b can be computed using the basis of (3.17) and (3.18). They read

$$R^\alpha_\beta = \int d\sigma \left(i[(P^Z)^\alpha Z_\beta - (P^Z)_\beta Z^\alpha] + \frac{i}{2} \delta^\alpha_\beta [(P^Z)_\gamma Z^\gamma - (P^Z)^\gamma Z_\gamma] \right. \\ \left. + \theta_\beta^\dagger \theta^\alpha - \theta^{\dagger\alpha} \theta_\beta + \frac{1}{2} \delta^\alpha_\beta [\theta^{\dagger\gamma} \theta_\gamma - \theta_\gamma^\dagger \theta^\gamma] \right) \quad (6.16)$$

$$L^a_b = \int d\sigma \left(i[(P^Y)^a Y_b - (P^Y)_b Y^a] + \frac{i}{2} \delta^a_b [(P^Y)_c Y^c - (P^Y)^c Y_c] \right. \\ \left. + \eta_b^\dagger \eta^a - \eta^{\dagger a} \eta_b + \frac{1}{2} \delta^a_b [\eta^{\dagger c} \eta_c - \eta_c^\dagger \eta^c] \right). \quad (6.17)$$

Using the above expressions for the supersymmetry generators, it is straightforward to compute their quantum anti-commutators. One indeed finds

$$\{Q^\alpha_a, \bar{Q}_\beta^b\} = \delta_a^b R^\alpha_\beta + \delta_\beta^\alpha L^b_a + \frac{1}{2} \delta_a^b \delta_\beta^\alpha H \quad (6.18)$$

with the Hamiltonian

$$H = 2 \int d\sigma \left[(P^Z)^\gamma (P^Z)_\gamma + (P^Y)^c (P^Y)_c + \frac{1}{4} (Z^\gamma Z_\gamma + Z'^\gamma Z'_\gamma + Y^c Y_c + Y'^c Y'_c) \right. \\ \left. + \frac{1}{2} (\theta^{\dagger\gamma} \theta_\gamma + \theta_\gamma^\dagger \theta^\gamma + \eta^{\dagger c} \eta_c + \eta_c^\dagger \eta^c) + \frac{1}{2} (\theta_\gamma^\dagger \theta^{\dagger\gamma} - \theta'^\gamma \theta_\gamma + \eta_c^\dagger \eta^{\dagger c} - \eta'^c \eta_c) - 2\delta(0) \right]. \quad (6.19)$$

The normal ordering contribution $-2\delta(0)$ of the fermions will cancel against the ground state energy of the bosons by supersymmetry upon introduction of creation and annihilation operators.

The $\mathfrak{su}(2)$ generators R^α_β and L^a_b (6.17) and (6.16) can be shown to obey the commutation relations

$$[R^\alpha_\beta, R^\gamma_\delta] = \delta_\beta^\gamma R^\alpha_\delta - \delta_\delta^\alpha R^\gamma_\beta, \quad [L^a_b, L^c_d] = \delta_b^c L^a_d - \delta_d^a L^c_b. \quad (6.20)$$

Next we turn to the quantum anticommutator $\{Q^\alpha_a, Q^\beta_b\}$ which evaluates to

$$\{Q^\alpha_a, Q^\beta_b\} = \frac{i}{2\zeta} \epsilon^{\alpha\beta} \epsilon_{ab} \int d\sigma e^{-ix_-} x'_- + \int d\sigma_1 d\sigma_2 \delta(\sigma_1 - \sigma_2) (\partial_{\sigma_1} + \partial_{\sigma_2}) \delta(\sigma_1 - \sigma_2)$$

We see that the potential last quantum anomaly cancels and we recover the central charge announced in (3.19) also at the quantum level. The analogous computation for $\{\bar{Q}^\alpha_a, \bar{Q}^\beta_b\}$ follows from conjugation.

Finally let us stress that in the above computations we have freely performed partial integrations by dropping contributions arising from the vertex operators $e^{\pm ix_-}$, as these would take us beyond the leading quadratic field approximation. These terms were dealt with, however, at the classical level up to order $\mathcal{O}(\zeta)$ as discussed in section four.

6.5 The $\mathfrak{su}(2|2)_R$ supercharge at quartic field order

Here we spell out the contribution to the right (blue) supercharges at quartic field order ($\mathcal{O}(\zeta^2)$) explicitly, restricting to the terms linear in fermions whose Poisson brackets yield the quartic bosonic Hamiltonian

$$\begin{aligned} Q^a_\alpha \Big|_{fbbb} = & \int d\sigma e^{-ix_-/2} \Big\{ \\ & (\theta^\alpha Y_a + \epsilon^{\alpha\beta} \epsilon_{ab} \theta_\beta Y^b) \left[-\frac{i}{4} (P^Y) \circ Y - \frac{1}{2} \mathcal{H}_{\text{bos}} \right] \\ & + \left(\theta^\alpha (2P^Y - iY)_a + \epsilon^{\alpha\beta} \epsilon_{ab} \theta_\beta (2P^Y - iY)^b \right) \left[\frac{i}{4} Y \circ Y \right] \\ & + (\theta^{\dagger\alpha} Y_a + \epsilon^{\alpha\beta} \epsilon_{ab} \theta^\dagger_\beta Y^b) \left[\frac{i}{4} (P^Y \circ Y' + P^Z \circ Z') + \frac{1}{4} Z \circ Z' - \frac{1}{8} Y \circ Y' \right] \\ & + (\theta^{\dagger\alpha} Y'_a + \epsilon^{\alpha\beta} \epsilon_{ab} \theta^\dagger_\beta Y'^b) \left[\frac{1}{2} Z \circ Z - \frac{1}{4} Y \circ Y \right] \\ & + (\eta^\dagger_a Z^\alpha + \epsilon^{\alpha\beta} \epsilon_{ab} \eta^{\dagger b} Z_\beta) \left[-\frac{1}{4} (P^Z) \circ Z + \frac{i}{2} \mathcal{H}_{\text{bos}} \right] \\ & + \left(\eta^\dagger_a (2P^Z + iZ)^\alpha + \epsilon^{\alpha\beta} \epsilon_{ab} \eta^{\dagger b} (2P^Z + iZ)_\beta \right) \left[\frac{1}{4} Z \circ Z \right] \\ & + (\eta_a Z^\alpha + \epsilon^{\alpha\beta} \epsilon_{ab} \eta^b Z_\beta) \left[\frac{1}{4} (P^Y \circ Y' + P^Z \circ Z') - \frac{i}{4} Y \circ Y' + \frac{i}{8} Z \circ Z' \right] \\ & + (\eta_a Z'^\alpha + \epsilon^{\alpha\beta} \epsilon_{ab} \eta^b Z'_\beta) \left[-\frac{i}{2} Y \circ Y + \frac{i}{4} Z \circ Z \right] \Big\}, \end{aligned} \quad (6.21)$$

where we have used the notation $(P^Z) \circ Z := (P^Z)_\gamma Z^\gamma + (P^Z)^\gamma Z_\gamma$ and $(P^Y) \circ Y := (P^Y)_c Y^c + (P^Y)^c Y_c$, etc. Also \mathcal{H}_{bos} denotes the bosonic part of the free (quadratic) Hamiltonian (6.19). Similar expressions follow for the left (red) supercharges.

6.6 The explicit form of the $\mathfrak{su}(2|2)_L$ generators

We denote all the generators appearing in the left (red) $\mathfrak{su}(2|2)_L$ algebra by lower case letters (with the exception of the common central charges). For the left (red)

supercharges we take as a basis

$$\begin{aligned}
q^1{}_1 &= \frac{1}{2} \text{str } Q \sigma^+ \otimes (\Gamma_{14} + \Gamma_{23} + \mathcal{P}_+) \\
q^2{}_2 &= -\frac{1}{2} \text{str } Q \sigma^+ \otimes (\Gamma_{14} + \Gamma_{23} - \mathcal{P}_+) \\
q^1{}_2 &= \text{str } Q \sigma^+ \otimes \Gamma_{12} \\
q^2{}_1 &= \text{str } Q \sigma^+ \otimes \Gamma_{34}.
\end{aligned} \tag{6.22}$$

One then finds at quadratic field order

$$\begin{aligned}
q^1{}_1 &= \frac{1}{2} \int d\sigma e^{ix-/2} \left[(2P^Z + iZ)^\gamma \theta_\gamma + iZ'^\gamma \theta_\gamma^\dagger + i(2P^Y - iY)_c \eta^{\dagger c} + Y'_c \eta^c \right] \\
q^2{}_2 &= \frac{1}{2} \int d\sigma e^{ix-/2} \left[(2P^Z + iZ)_\gamma \theta^\gamma + iZ'_\gamma \theta^{\dagger \gamma} + i(2P^Y - iY)^c \eta_c^\dagger + Y'^c \eta_c \right] \\
q^1{}_2 &= \frac{1}{2} \int d\sigma e^{ix-/2} \left[\epsilon_{\alpha\beta} \left((2P^Z + iZ)^\alpha \theta^\beta + iZ'^\alpha \theta^{\dagger \beta} \right) - \epsilon_{ab} \left(i(2P^Y - iY)^a \eta^{\dagger b} + Y'^a \eta^b \right) \right] \\
q^2{}_1 &= -\frac{1}{2} \int d\sigma e^{ix-/2} \left[\epsilon^{\alpha\beta} \left((2P^Z + iZ)_\alpha \theta_\beta + iZ'_\alpha \theta_\beta^\dagger \right) - \epsilon^{ab} \left(i(2P^Y - iY)_a \eta_b^\dagger + Y'_a \eta_b \right) \right]
\end{aligned} \tag{6.23}$$

and their complex conjugated partners $\bar{q}_A{}^B$. These generators can be shown to anti-commute with the right (blue) supercharges $Q^\alpha{}_a$ and $\bar{Q}_\alpha{}^a$, their commutation with the right (blue) $\mathfrak{su}(2)$ generators $R^\alpha{}_\beta$ and $L^a{}_b$ is manifest due to the (right) covariant notation.

Translating these charges into the barred coordinates with the flipped “2” index of (6.14) enables one to write the left (red) supercharges covariantly

$$\begin{aligned}
q^A{}_B &= \frac{1}{2} \int d\sigma e^{ix-/2} \left[(2\bar{p}^Z + i\bar{Z})^A \bar{\theta}_B + i\bar{Z}'^A \bar{\theta}_B^\dagger + i(2\bar{p}^Y - i\bar{Y})_B \bar{\eta}^{\dagger A} + \bar{Y}'_B \bar{\eta}^A \right. \\
&\quad \left. + \epsilon^{AC} \epsilon_{BD} \left((2\bar{p}^Z + i\bar{Z})_C \bar{\theta}^D + i\bar{Z}'_C \bar{\theta}^{\dagger D} + i(2\bar{p}^Y - i\bar{Z})^D \bar{\eta}_C^\dagger + \bar{Y}'^D \bar{\eta}_C \right) \right]
\end{aligned} \tag{6.24}$$

and the complex conjugate expression $\bar{q}_A{}^B$. Here we note the conjugation properties $(\bar{Z}_A)^\dagger = \bar{Z}^A$, $(\bar{\theta}_A)^\dagger = \bar{\theta}^{\dagger A}$, $(\bar{p}_A^Z)^\dagger = (\bar{p}^Z)^A$, etc. One then computes the anti-commutator

$$\{q^A{}_B, \bar{q}_C{}^D\} = -\delta_C^A l_B^D - \delta_B^D r_C^A + \frac{1}{2} \delta_C^A \delta_B^D H \tag{6.25}$$

with the same quadratic Hamiltonian H appearing in the $\mathfrak{su}(2|2)_L$ algebra. The

bosonic $\mathfrak{su}(2)$ generators appearing on the right hand side are given by

$$\begin{aligned} r^A{}_B = i \left[(\bar{p}^Z)^A \bar{Z}_B - (\bar{p}^Z)_B \bar{Z}^A \right] + \frac{i}{2} \delta_B^A \left[(\bar{p}^Z)_C \bar{Z}^C - (\bar{p}^Z)^C \bar{Z}_C \right] \\ - \left[(\bar{\eta}^\dagger)^A \bar{\eta}_B - (\bar{\eta}^\dagger)_B \bar{\eta}^A \right] + \frac{1}{2} \delta_B^A \left[(\bar{\eta}^\dagger)^C \bar{\eta}_C - (\bar{\eta}^\dagger)_C \bar{\eta}^C \right], \end{aligned} \quad (6.26)$$

$$\begin{aligned} l^A{}_B = i \left[(\bar{p}^Y)^A \bar{Y}_B - (\bar{p}^Y)_B \bar{Y}^A \right] + \frac{i}{2} \delta_B^A \left[(\bar{p}^Y)_C \bar{Y}^C - (\bar{p}^Y)^C \bar{Y}_C \right] \\ - \left[(\bar{\theta}^\dagger)^A \bar{\theta}_B - (\bar{\theta}^\dagger)_B \bar{\theta}^A \right] + \frac{1}{2} \delta_B^A \left[(\bar{\theta}^\dagger)^C \bar{\theta}_C - (\bar{\theta}^\dagger)_C \bar{\theta}^C \right]. \end{aligned} \quad (6.27)$$

They are traceless and obey the $\mathfrak{su}(2)$ algebra.

Finally one again computes the anti-commutator

$$\{q^A{}_B, q^C{}_D\} = -\frac{i}{2} \epsilon^{AC} \epsilon_{BD} \int d\sigma e^{i x_-} \left[(P^Z) \circ Z' + (P^Y) \circ Y' + i(\theta^\dagger \circ \theta' + \eta^\dagger \circ \eta') \right]$$

giving rise to the level-matching condition as we had in the right (blue) algebra. We note that the right-hand side of the above takes the same form in barred or unbarred variables.

6.7 The centrality of the level-matching and Hamiltonian

In this section, we show that the level-matching generator p_{ws} and the Hamiltonian do Poisson-commute with all the generators of the $\mathfrak{su}(2|2) \oplus \mathfrak{su}(2|2)$ algebra. The explicit computation follows the logic of section 4.

Since we would like to work in the limit of infinite P_+ we also have to suppress the corresponding conjugate zero mode x_- . We pick up a solution for the unphysical field $x_-(s)$ which obeys the boundary condition $x_-(-\infty) = 0$. It reads as

$$x_-(s) = -\zeta \int_{-\infty}^s d\omega \left(p_M x'_M - \frac{i}{2} \text{str}(\Sigma_+ \chi \chi') \right).$$

Using the canonical Poisson brackets it is easy to find

$$\begin{aligned} \frac{1}{\zeta} \{p_M(\sigma), x_-(s)\} &= \delta(\sigma - s) p_M(\sigma) - p'_M(\sigma) \epsilon(s - \sigma), \\ \frac{1}{\zeta} \{x_M(\sigma), x_-(s)\} &= -x'_M(\sigma) \epsilon(s - \sigma), \\ \frac{1}{\zeta} \{x'_M(\sigma), x_-(s)\} &= -x''_M(\sigma) \epsilon(s - \sigma) + x'_M(\sigma) \delta(\sigma - s), \\ \frac{1}{\zeta} \{\chi(\sigma), x_-(s)\} &= \frac{1}{2} \delta(\sigma - s) \chi(\sigma) - \chi'(\sigma) \epsilon(s - \sigma). \end{aligned} \quad (6.28)$$

Here $\epsilon(s)$ is the standard step function

$$\epsilon(s) = \begin{cases} 1, & s \geq 0, \\ 0, & s < 0 \end{cases}, \quad (6.29)$$

which satisfies the condition $\epsilon(s) + \epsilon(-s) = 1$. The reader can easily verify the validity of these formulae by considering, e.g., the Jacobi identity.

First, using these formulae one can check that the supercharges commute with the level-matching generator. Introducing the level-matching generator

$$p_{\text{ws}} = -\zeta \int_{-\infty}^{\infty} d\omega \left(p_M x'_M - \frac{i}{2} \text{str}(\Sigma_+ \chi \chi') \right)$$

it is easy to see that

$$\frac{1}{\zeta} \{p_{\text{ws}}, x_M(s)\} = x'_M(s), \quad \frac{1}{\zeta} \{p_{\text{ws}}, p_M(s)\} = p'_M(s), \quad \frac{1}{\zeta} \{p_{\text{ws}}, \chi(s)\} = \chi'(s)$$

and, therefore,

$$\frac{1}{\zeta} \{p_{\text{ws}}, x_-(s)\} = -\zeta \left(p_M x'_M - \frac{i}{2} \text{str}(\Sigma_+ \chi \chi') \right)(s) = x'_-(s).$$

Thus,

$$\frac{1}{\zeta} \{p_{\text{ws}}, Q\} = \int \partial_s Q = 0$$

provided all the fields rapidly decrease at infinity.

Note that $x_-(s)$ is quadratic in fermions, while we are interested in the contribution to the Poisson bracket of H and Q which is linear in fermions. This observation implies that computing the Poisson bracket of the Hamiltonian with $e^{i\alpha x_-(s)}$ it is enough to use instead of the full H the quadratic bosonic Hamiltonian with the density $\mathcal{H}_2^b(\sigma)$. Using the basic Poisson brackets it is easy to find

$$\frac{1}{\zeta} \{\mathcal{H}_2^b(\sigma), x_-(s)\} = (p_M^2 + x_M'^2) \delta(\sigma - s) - \partial_\sigma \mathcal{H}_2^b \epsilon(s - \sigma).$$

Finally, to verify the centrality of the Hamiltonian up to order ζ we have to compute

$$\begin{aligned} \{H, Q\} = \int \int d\sigma ds & \left[\{\mathcal{H}_2^b, e^{i\alpha x_-}\} \Omega_2 + \right. \\ & \left. + e^{i\alpha x_-} (\{\mathcal{H}_2, \Omega_2\} + \zeta \{\mathcal{H}_4, \Omega_2\} + \zeta \{\mathcal{H}_2, \Omega_4\}) \right] + \dots \end{aligned}$$

Here the integrals are taken from $-\infty$ to $+\infty$ and to simplify the notation we do not exhibit the dependence of functions Ω on physical fields. We have

$$\begin{aligned} \{H, Q\} = i\alpha\zeta \int ds & e^{i\alpha x_-} (p_M^2 + x_M'^2 - \mathcal{H}_2^b) \Omega_2 \\ & + \int \int d\sigma ds e^{i\alpha x_-} (\{\mathcal{H}_2, \Omega_2\} + \zeta \{\mathcal{H}_4, \Omega_2\} + \zeta \{\mathcal{H}_2, \Omega_4\}). \end{aligned}$$

The further computation is straightforward and it uses explicit expressions for $\Omega_{2,4}$ in terms of transverse fields. We note that the Poisson bracket

$$\{\mathcal{H}_2, \Omega_2\} + \zeta\{\mathcal{H}_4, \Omega_2\} + \zeta\{\mathcal{H}_2, \Omega_4\} \quad (6.30)$$

contains terms proportional to $\delta(\sigma - s)$, $\delta'(\sigma - s)$ and $\delta''(\sigma - s)$ which reduces the double integration to a single one. Moreover, according to our assumptions about the orders of perturbation theory we are working on, in the expression (6.30) only the terms linear in fermions should be taken into account. This means, in particular, that in this specific computation only the terms in \mathcal{H}_4 which are quadratic in fermions matter. Evaluating the brackets under these assumptions we find that up to the order ζ the integrand appears to be a total derivative and therefore vanishes for fields with rapidly decreasing boundary conditions. Thus, with our assumptions we have verified that

$$\{H, Q\} = 0,$$

i.e. the Hamiltonian commutes with all dynamical supercharges. It is not difficult to extend this treatment to higher orders in fermions and in ζ but it is already clear that we will not find any anomaly because of a rigid structure of the supersymmetry algebra: the complete Hamiltonian will commute with all dynamical supercharges.

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